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LINEAR AND NONLINEAR PROGRAMMING METHODS IN OPTIMAL RESOURCE ALLOCATION PROBLEMS

Qodirova D. D
Student, group ST-30/25
Tashkent State University of Economics
Uzbekistan, Tashkent
Soliyev J. K
Student, group ST-30/25
Tashkent State University of Economics
Uzbekistan, Tashkent
Poshakhodjaeva G. D
Associate Professor
Tashkent State University of Economics
Uzbekistan, Tashkent

Abstract: This scientific work examines linear and nonlinear programming methods used to solve optimal allocation problems for limited resources among competing activities. Classical problem formulations are analyzed, along with the applicability conditions of the simplex method and nonlinear optimization methods (Lagrange multiplier method, Frank-Wolfe method). Three detailed practical examples are provided: optimal budget allocation of an investment fund among production projects, an optimal production plan for an industrial enterprise, and optimal advertising budget allocation with a nonlinear response function.

Keywords: linear programming, nonlinear programming, optimal resource allocation, simplex method, objective function.

Introduction and Problem Statement

The problem of optimal allocation of limited resources is one of the central issues in economic planning. An enterprise, possessing finite amounts of capital, labor resources, and production capacities, seeks to distribute them in such a way as to maximize output or profit while satisfying all technological and financial constraints.

Depending on the nature of the objective function and constraints, resource allocation problems are divided into two classes: linear programming (LP) problems — when the objective function and all constraints are linear, and nonlinear programming (NLP) problems — when at least one of these elements is nonlinear. Both classes are widely used in production management, financial planning, and logistics.

Linear Programming in Resource Allocation Problems

A standard linear programming problem in the context of resource allocation is formulated as follows. Suppose there are n types of activities (projects, products, or investment directions) and m types of resources. The volume of the j-th activity is denoted by

x_j. It is required to maximize the linear objective function:

$F(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow max,$

where c_j -is the unit contribution (revenue or profit) of one unit of the j- th activity, subject to resource constraints:

$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i \ (i = 1, 2, \ldots, m), \qquad (1)$

where a_{ij} - is the amount of the *i*- th resource required per unit of the j- th activity, b_i - is the total availability of the *i*- th resource, together with the non-negativity conditions:

$x_j \geq 0 \ (j = 1, 2, \ldots, n). \qquad (2)$

To solve problem (1)–(2), the simplex method is applied, which is based on examining the vertices of the feasible polyhedron. The optimality theorem states that if the feasible set is nonempty and bounded, then the optimum is attained at one of its corner points.

An important property of linear programming problems is duality. Every LP problem has a corresponding dual problem whose solution provides the shadow prices of resources. Their economic interpretation lies in the marginal increase of the objective function resulting from an increase of one unit in the availability of the corresponding resource.

Nonlinear Programming in Resource Allocation Problems

In a number of economic problems, the assumption of linearity is violated: the return on investment decreases as the amount of investment increases (diminishing returns to scale), the cost of a resource depends on the purchase volume, or the risk function is nonlinear. In such cases, a nonlinear programming problem is formulated:

$F(x_1, x_2, \ldots, x_n) \rightarrow max \text{ (or min)},$

$g_i(x_1, x_2, \ldots, x_n) \leq b_i \ (i = 1, 2, \ldots, m), \qquad (3)$

$x_j \geq 0 \ (j = 1, 2, \ldots, n),$

where F and g_i - are differentiable functions that are not necessarily linear.

If the problem contains equality constraints and the functions are differentiable, the method of Lagrange multipliers is used to determine the optimum. The Lagrangian function is constructed as follows:

$L(x, \lambda) = F(x) - \Sigma_i \lambda_i \cdot g_i(x), \qquad (4)$

where λ_i -are the Lagrange multipliers (shadow prices of the constraints). The first-order conditions ($\partial L/\partial x_j = 0$, $\partial L/\partial \lambda_i = 0$) form a system of equations whose solution determines the optimal point. For problems with inequality constraints, the Kuhn–Tucker conditions serve as a generalization.

When the objective function and the constraints are convex, every local optimum is also a global optimum. This is a key condition for the applicability of gradient-based methods (such as the Frank–Wolfe method and the projected gradient method) to nonlinear programming problems in economics.

Practical Example 1: Optimal Allocation of an Investment Budget

Consider the following problem. An investment fund has a budget of B = 500 million Uzbek soums. The funds are to be allocated among three production projects. For each project, the expected return per unit of investment, labor resource requirements, and production capacity requirements are known. The data are summarized in Table 1

Table 1. Parameters of Investment Projects

Parameter	Project 1 (Machine Tool Manufacturing)	Project 2 (Agricultural Processing)	Project 3 (IT Development)
Return (million soums per million soums invested)	0,28	0,19	0,35
Labor intensity (person-days per million soums)	0,40	0,55	0,20
Capacity utilization (units per million soums)	0,30	0,25	0,10
Minimum investment volume (million soums)	50	30	20
Maximum investment volume (million soums)	250	180	200

Let the investment amounts in projects 1, 2 and 3 be denoted by x_1, x_2, x_3 (in million soums). The linear programming problem is formulated as follows:

$F = 0{,}28x_1 + 0{,}19x_2 + 0{,}35x_3 \rightarrow \max,$

$x_1 + x_2 + x_3 \leq 500,$ (5)

$0{,}40x_1 + 0{,}55x_2 + 0{,}20x_3 \leq 220,$ (6)

$0{,}30x_1 + 0{,}25x_2 + 0{,}10x_3 \leq 140,$ (7)

$50 \leq x_1 \leq 250;\ 30 \leq x_2 \leq 180;\ 20 \leq x_3 \leq 200.$ (8)

Applying the simplex method to system (5)–(8), we obtain the optimal solution (Table 2).

Table 2. Optimal Budget Allocation Plan

Indicator	Project 1	Project 2	Project 3	Total
Optimal volume (million soums)	50	30	200	280
Expected income (million soums)	14,0	5,7	70,0	89,7

Labor intensity (person-days)	20,0	16,5	40,0	76,5
Capacity (units)	15,0	7,5	20,0	42,5

The optimal plan prescribes investing the minimum allowable amounts in Projects 1 and 2 (as less profitable ones) and allocating the maximum allowable amount to Project 3 (IT Development), which has the highest return of 0.35. The remaining budget (220 million soums) remains unused. The maximum total profit amounts to 89.7 million soums.

Analysis of the dual variables shows that the shadow price of the labor resource constraint (6) is zero—this resource is not scarce. The shadow price of the budget constraint (5) is 0.35 million soums per million soums: each additional million soums of budget increases profit by 0.35 million soums.

Practical Example 2: Optimal Production Plan of an Enterprise

Consider the problem of optimal production planning at an industrial enterprise. The factory produces four types of products (P1–P4) using three types of resources: machine time (not more than 480 machine-hours per shift), labor resources (not more than 350 man-hours), and raw materials (not more than 600 kg). The resource consumption norms and profit per unit of output are presented in Table 3.

Table 3. Production Parameters of the Enterprise

Resource / Indicator	**P1 (Machinery)**	**P2 (Gear)**	**P3 (Housing)**	**P4 (Flange)**	**Limit**
Machine time (machine-hours/unit)	2,0	3,5	1,5	4,0	≤ 480
Labor (man-hours/unit)	1,5	2,0	1,0	3,0	≤ 350
Raw materials (kg/unit)	4,0	6,0	3,0	5,0	≤ 600
Profit (thousand soums/unit)	18	30	12	40	—
Minimum output (units)	20	10	15	0	—
Maximum output (units)	80	60	100	50	—

Let the output levels of products be denoted by x_1, x_2, x_3, x_4 (units). The linear programming problem is formulated as follows:

$F = 18x_1 + 30x_2 + 12x_3 + 40x_4 \rightarrow \max,$

$2{,}0x_1 + 3{,}5x_2 + 1{,}5x_3 + 4{,}0x_4 \leq 480,$ (9)

$1{,}5x_1 + 2{,}0x_2 + 1{,}0x_3 + 3{,}0x_4 \leq 350,$ (10)

$4{,}0x_1 + 6{,}0x_2 + 3{,}0x_3 + 5{,}0x_4 \leq 600,$ (11)

$20 \leq x_1 \leq 80;\ 10 \leq x_2 \leq 60;\ 15 \leq x_3 \leq 100;\ 0 \leq x_4 \leq 50.$ (12)

The simplex method yields an optimal plan, presented in Table 4.

Table 4. Optimal Production Plan

Indicator	P1 (Machinery)	P2 (Gear)	P3 (Housing)	P4 (Flange)	Total
Optimal output (units)	20	10	15	50	95
Profit (thousand soums)	360	300	180	2 000	2 840
Machine time (machine-hours)	40	35	22,5	200	297,5
Labor (man-hours)	30	20	15	150	215
Raw materials (kg)	80	60	45	250	435

The optimal solution prescribes producing P1, P2, and P3 at their minimum allowable levels, while P4 (Flange) is produced at its maximum possible level (50 units), since it provides the highest profit of 40 thousand soums per unit. The total profit amounts to 2,840 thousand soums per shift. All resource constraints are satisfied with slack, which indicates zero shadow prices for constraints (9)–(11) at the given solution. The binding constraint is the upper production limit for the flange (12): increasing it by 1 unit results in an increase in profit of exactly 40 thousand soums.

Practical Example 3: Nonlinear Advertising Budget Allocation Problem

Nonlinear programming problems are particularly relevant in marketing, where the return on advertising investment follows the law of diminishing returns: initial investments in a channel produce the highest impact, while each additional unit of spending yields a progressively smaller increase in audience reach.

Consider the problem of allocating an advertising budget B = 300 million soums across four channels: television (x_1), online advertising (x_2), outdoor advertising (x_3) and radio (x_4). The reach function for each channel is approximated by a concave function of the form $\alpha_j\sqrt{x_j}$, reflecting diminishing marginal returns. The model parameters are presented in Table 5.

Table 5. Advertising Channel Parameters

Parameter	TV (x_1)	Internet (x_2)	Outdoor (x_3)	Radio (x_4)
Reach coefficient α_j (thousand people / √million soums)	85	110	60	45
Minimum investment (million soums)	30	20	10	0
Maximum investment (million soums)	150	120	80	60
Cost per contact (soums per person)	12	8	15	6

The objective function is to maximize the total audience reach:

$F(x) = 85\sqrt{x_1} + 110\sqrt{x_2} + 60\sqrt{x_3} + 45\sqrt{x_4} \to \max,$ (13)

subject to the following constraints:

$x_1 + x_2 + x_3 + x_4 \leq 300,$ (14)

$30 \leq x_1 \leq 150;\ 20 \leq x_2 \leq 120;\ 10 \leq x_3 \leq 80;\ 0 \leq x_4 \leq 60.$ (15)

The function *F* is concave (a sum of concave functions), and the constraints are convex; therefore, the problem belongs to the class of convex programming problems. We apply the Kuhn–Tucker conditions. The Lagrangian function is:

$L = 85\sqrt{x_1} + 110\sqrt{x_2} + 60\sqrt{x_3} + 45\sqrt{x_4} - \lambda(x_1 + x_2 + x_3 + x_4 - 300).$ (16)

The optimality condition for the active constraint (14) when $x_j > 0$ is:

$\partial L/\partial x_j = \alpha_j / (2\sqrt{x_j}) - \lambda = 0 \implies x_j^* = (\alpha_j / 2\lambda)^2.$ (17)

Substituting into the budget constraint $x_1 + x_2 + x_3 + x_4 = 300$ and taking into account the upper bounds (15), we obtain, using an iterative method $\lambda^* \approx 5{,}62$. The optimal solution is presented in Table 6.

Table 6. Optimal Allocation of the Advertising Budget
*** one channel has reached its upper bound; λ is recalculated taking this constraint into account**

Indicator	TV (x_1)	Internet (x_2)	Outdoor (x_3)	Radio (x_4)	Total
Optimal investments (million soums)	114,5	120,0*	28,5	37,0	300,0

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Audience reach (thousand people)	909	1 206	320	274	2 709
Cost per reach (soums per person)	126	99	89	135	111

Online advertising (x_2) reaches its upper limit of 120 million soums, since its reach coefficient α_2 = 110 s the highest; this channel is therefore the most efficient in terms of return per unit of cost. The total audience reach amounts to 2,709 thousand people, with an average unit cost of 111 soums per contact.

The shadow price of the budget constraint $\lambda^* \approx 5{,}62$ thousand people per million soums implies that each additional million soums of advertising budget increases audience reach by approximately 5,620 people. This value can serve as a guideline for justifying an expansion of the marketing budget.

Nonlinear Extension of the Investment Problem

In reality, investment returns typically decrease as the volume of investment increases due to market saturation. Suppose that the revenue function of Project 3 (Example 1) has the following form:

$$f_3(x_3) = 0{,}35x_3 - 0{,}0005x_3^2, \qquad (18)$$

which corresponds to diminishing marginal returns when $x_3 > 350$. Then the problem becomes nonlinear and the objective function takes the form:

$$F = 0{,}28x_1 + 0{,}19x_2 + 0{,}35x_3 - 0{,}0005x_3^2 \rightarrow \max. \qquad (19)$$

The function (19) is concave, and the constraints (5)–(8) are convex; therefore, the problem is a convex programming problem, and the Kuhn–Tucker conditions are both necessary and sufficient. Applying the Frank–Wolfe method (successive linearization), we obtain Corrected optimum: $x_1 = 50$, $x_2 = 30$, $x_3 = 200$. In this range, the nonlinear correction is small ($-0{,}0005 \cdot 200^2 = -20$ million soums), and the adjusted profit amounts to 79.7 million soums. If the constraint $x_3 \leq 200$ is removed, the nonlinear problem yields an optimum x_3^* = million soums, with a profit of 61.25 million soums for Project 3.

Conclusions

Methods of linear and nonlinear programming provide a powerful toolkit for solving problems of optimal resource allocation in economics. The three examples considered demonstrate the universality of this approach: the allocation of an investment fund's budget, the optimization of an enterprise's production plan, and a nonlinear advertising planning problem are all solved within a unified formal framework, despite differences in the structure of the objective function and constraints.

Linear programming is effective under linear relationships and allows for a rich economic interpretation through duality theory: the shadow prices of resources make it possible to assess the economic value of each constraint and to justify decisions on investing in the expansion of production capacities.

Nonlinear programming generalizes these approaches to the case of diminishing returns, nonlinear costs, and quadratic risk functions. The key condition for applicability is convexity of the problem, which guarantees that any obtained optimum is global. The combined use of both classes of methods makes it possible to construct realistic and computationally tractable models for the allocation of production, financial, and marketing resources.

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